Lecture 3 & 4.

- Complex tori
- Line bundles on a complex torus
- Algebraizability of $J(C)$.

<u>Reference</u>: Mumford, Abelian Varieties, 1970.

§1. Complex tori

<u>Def</u> A <u>complex torus</u> X is a compact, connected complex Lie group.

so, the multiplication map

$$m : X \times X \longrightarrow X$$

and the inverse map

$$\iota : X \longrightarrow X$$

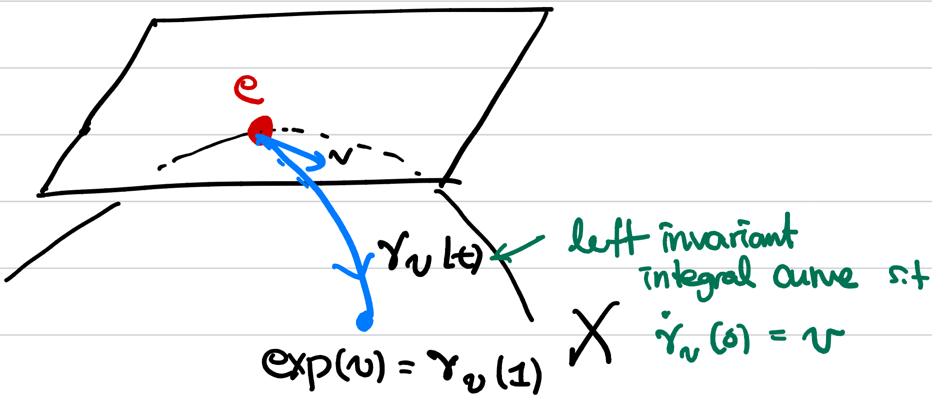
are holomorphic. $\dim_{\mathbb{C}} X = g$.

Local geometry of X is determined by the geometry near $e \in X$

$$\begin{tikzcd} g^{-1} : X \arrow[r, "\cong"] & X \\[-20pt] \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} \\[-20pt] g \arrow[r, mapsto] & e \end{tikzcd}$$

Let $V = T_eX$: tangent space at the identity e.
We have the exponential map

$$\exp : T_eX \longrightarrow X$$



Check
- $\exp$ is a surjective group homomorphism
- $\ker(\exp) =: \Gamma \subset V$ is a full lattice ($\cong \mathbb{Z}^{2g}$).
- $\exp : V \to X$ is the universal cover.

$\Rightarrow X \cong V/\Gamma$ ← this explains the notation

example
- $X = J(C) = H^0(K_C)^\vee / H_1(C, \mathbb{Z})$
- $X = \mathrm{Pic}^0(C)$

□ Singular (co)homology

$X = \mathbb{C}^g/\Gamma \Rightarrow X \simeq_{homeo} (S^1)^{2g}$.

$\pi : V \longrightarrow X$: univ. cover $\Rightarrow \pi_1(X,e) \cong \Gamma$

Since Γ is abelian, $H_1(X,\mathbb{Z}) \cong \Gamma$ and

$$H^1(X,\mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(H_1(X,\mathbb{Z}),\mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(\Gamma,\mathbb{Z}).$$

We have cup product map

$$\Lambda^p H^1(X,\mathbb{Z}) \longrightarrow H^p(X,\mathbb{Z})$$

which is an isom (because $X \simeq (S^1)^{2g}$). Hence

$$H^p(X,\mathbb{Z}) \simeq \Lambda^p \mathrm{Hom}(\Gamma,\mathbb{Z}).$$

§2. Outline

X : complex torus of dim $= g$.

Q) When X is a projective variety ?

$$X \hookrightarrow \mathbb{P}^N_{\mathbb{C}} \qquad X = V(F_1, \cdots F_m) ?$$

- When $g \geq 2$, general complex torus is NOT a proj variety.
- By Theorem of Chow, any closed analytic subset of $\mathbb{P}^N_{hol}$ is Zariski closed in $\mathbb{P}^N_{hol}$.

This question is related to find a nice (very ample) line bundle L on X: $s_0, \cdots s_N \in H^0(X, L)$

$$\varphi_L : X \dashrightarrow \mathbb{P}^N_{\mathbb{C}} \qquad x \longmapsto [s_0(x) : \cdots : s_N(x)]$$

Any compact, holomorphic manifold has at most one algebraic structure.
(not true for non compact holo. manifold)

When X is a $\mathbb{C}$-torus.

X: alg variety $\Longleftrightarrow$ X: projective variety.

PLAN:

(i) line bundles on X, sections?

(ii) properties of L which make φ_L: closed imbedding

(iii) Riemann's bilinear relations & positivity

$\rightsquigarrow$ line bundle on $J(C)$

$\rightsquigarrow$ prove $J(C) \hookrightarrow \mathbb{P}^N_{\mathbb{C}}$.

Digression : non-algebraicity of general tori.

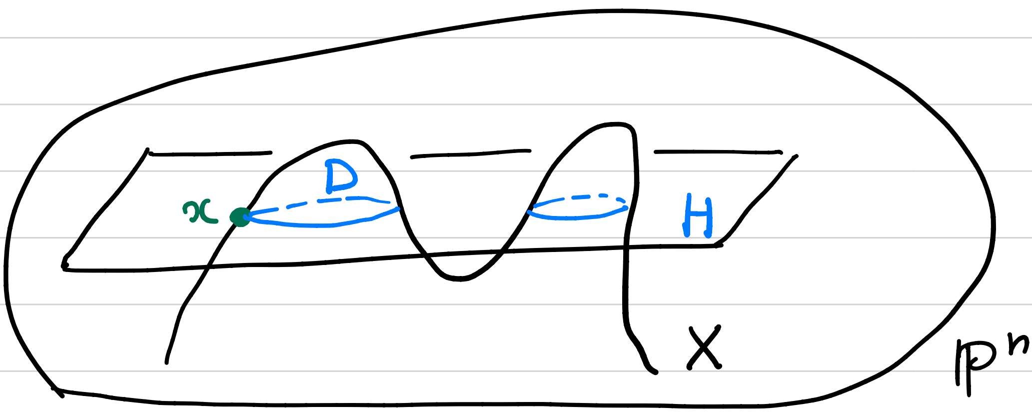
Nice feature of $X \subset \mathbb{P}^N$: We have enough hyperplanes !

Lemma Let $X^n \subset \mathbb{P}^N$ be a nonsingular proj var. Then $\exists$ nonsingular divisor $D \subset X$ s.t

$$[D] \in H_{2n-2}(X, \mathbb{Z})$$

is nonzero.

Pf) Let $x \in X$ and $H \subset \mathbb{P}^N$: hyperplane passing through x.



For general H, $D := H \cap X$: nonsingular. (Theorem of Bertini)

Similarly, for general $(N-n+1)$ plane L through x,

$$C = L \cap X$$

is a nonsingular curve on X which intersect D transversely. ie $T_x C \oplus T_x D = T_x X$.

Nontriviality of D can be captured by the Intersection product.

$$[D].[C] \geq \#(C \cap D) > 0$$

$\Rightarrow [D] \neq 0$ $\blacksquare$

Let $X = \mathbb{C}^2/\Gamma$. 2 dim'l $\mathbb{C}$-torus

Choose a basis $\{\alpha, \beta, \gamma, \delta\}$ of $\Gamma = H_1(X, \mathbb{Z})$.

$$\Rightarrow \quad H_2(X, \mathbb{Z}) = \mathbb{Z}\langle \alpha\wedge\beta,\ \alpha\wedge\gamma,\ \alpha\wedge\delta,\ \beta\wedge\gamma,\ \beta\wedge\delta,\ \gamma\wedge\delta \rangle$$

Let $dz_1, dz_2 \in H^1(X, \mathbb{C})$: holomorphic 1 forms on X.

$$\int_\alpha dz_1 = \alpha_1 \quad \& \quad \int_\alpha dz_2 = \alpha_2 .$$

Let $P = \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 \end{pmatrix}$

$\boxed{\text{"period matrix"}}$

encodes complex structure on X

$\omega = dz_1 \wedge dz_2 \in H^2(X, \mathbb{C})$: holomorphic 2-form on X.

<u>Check</u> $\displaystyle\int_{[\alpha \wedge \beta]} \omega = \det \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix} \neq 0$.

so the action

$$\omega \cap : H_2(X, \mathbb{Z}) \longrightarrow \mathbb{C}$$

is determined by the minors of P.

For general choice of Γ, minors of P is $\mathbb{Z}$ - linearly independent. ie if $D \in H_2(X, \mathbb{Z})$,

$$\int_D \omega = 0 \iff D = 0.$$

If X is projective, $\exists$ nonsingular curve $D \subset X$ s.t $[D] \neq 0$. However

$$\int_D \omega = 0 \quad \text{bc} \quad \omega : \text{holomorphic 2-form.}$$

$\therefore$ For general Γ, $\mathbb{C}^2/\Gamma$ is not projective.

(it really depends on the complex structure of $\mathbb{C}^2/\Gamma$) $\square$

§3 Group cohomology.

* G : discrete group, M : $\mathbb{Z}[G]$-module.
(abelian group with G-action)

Let

$$(\ )^G : \mathbb{Z}[G]\text{-Mod} \to Ab$$

defined by

$$M^G = \{ m \in M \mid g.m = m \ \forall g \in G \}.$$

$(-)^G$ is a left exact functor.

<u>Def</u> $H^p(G.M) := H^p(R(M)^G)$.

As a chain complex:

$C^p(G.M) = \{ f : G^p \longrightarrow M \}$

$\delta : C^p(G.M) \longrightarrow C^{p+1}(G.M)$

$$f \longmapsto (\delta f)(\sigma_0 \cdots \sigma_{p+1}) = \sigma_0 f(\sigma_1, \cdots, \sigma_p)$$

$$+ \sum_{i=0}^{p-1} (-1)^{i+1} f(\sigma_0, \cdots, \overbrace{\sigma_i \sigma_{i+1}}^{\text{multiplied}}, \cdots \sigma_p)$$

$$+ (-1)^{p+1} f(\sigma_0, \cdots, \sigma_{p-1}).$$

Then

$$H^p(G.M) = H^p(C^\bullet(G.M), \delta).$$

Eg. • $H^0(G.M) = M^G$

• $H^1(G.M) = \dfrac{\{f : G \to M \mid f(ab) = f(a) + af(b)\}}{\{f : G \to M \mid f(a) = am - m, \exists m \in M\}}.$

If $G \curvearrowright M$ trivial, $H^1(G.M) = \mathrm{Hom}(G.M)$.

group homomorphisms

Let Y : Hausdorff space $\circlearrowleft G$. freely, discontinuously
(ie $\forall y \in Y$, $\exists U_y \ni y$ st $U_y \cap g(U_y) = \phi$ $\forall g \neq e$).

$\pi : Y \longrightarrow Y/G =: X$. quotient map

<u>Prop</u> For any sheaf of abelian groups $\mathcal{F}$ on X, there exists a morphism

$$\phi : H^p(G, H^0(Y, \pi^{-1}\mathcal{F})) \longrightarrow H^p(X, \mathcal{F}).$$

If $H^i(Y, \pi^{-1}\mathcal{F}) = 0$ $\forall i > 0$, then ϕ is an isom.
↳ Usual Grothendieck spectral seq argument.

<u>Eg</u> Let $X = V/\Gamma$. Since V is contractible, $H^{>0}(V, \mathbb{Z}) = 0$
and

$$\begin{array}{ccc} H^1(\Gamma, \mathbb{Z}) & \xrightarrow[\cong]{\phi} & H^1(X, \mathbb{Z}) \\ \parallel & & \parallel \\ \mathrm{Hom}(\Gamma, \mathbb{Z}) & & \mathrm{Hom}(H_1(X, \mathbb{Z}), \mathbb{Z}) = \mathrm{Hom}(\Gamma, \mathbb{Z}) \end{array}$$

§ Line bundles on $X = \mathbb{C}^g/\Gamma$.

BLACKBOX $H^p(\mathbb{C}^N, \mathcal{O}_{\mathbb{C}^N}) = 0 \quad \forall p > 0$.

↳ In particular any holomorphic line bundles on $\mathbb{C}^N$ are trivial.

Let $L \in H^1(X, \mathcal{O}_X^*)$.

$$\chi : \pi^* L \xrightarrow{\cong} V \times \mathbb{C}$$
$$\circlearrowleft \Gamma \qquad\qquad \circlearrowleft \Gamma$$

s.t. $L \cong V \times \mathbb{C}/\Gamma \xrightarrow{\pi} V/\Gamma = X$

$$(z, \lambda) \longmapsto (z+\gamma, e_\gamma(z) \cdot \lambda) \qquad \gamma \in \Gamma.$$

for some $e_\gamma \in H^0(V, \mathcal{O}_V^*)$.

- Group action $\Rightarrow$ $e_{\gamma+\gamma'}(z) = e_\gamma(z+\gamma') e_{\gamma'}(z)$.

In order to understand line bundles :

$$H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}).$$

V : simply connected $\Rightarrow$

$$0 \to \mathbb{Z} \to \underset{\substack{\parallel \\ H}}{H^0(V, \mathcal{O}_V)} \xrightarrow{e^{2\pi i}} \underset{\substack{\parallel \\ H^*}}{H^0(V, \mathcal{O}_V^*)} \to 0$$

exact. and we have a diagram

$$\begin{tikzcd}
 & F & E \arrow[l, mapsto] \\
\{e_\gamma\} \in H^1(\Gamma, H^*) \arrow[ur, bend left, mapsto] \arrow[r, "\delta"] \arrow[d, "\phi"', "\cong"] \arrow[dr, phantom, "\circlearrowright"] & H^2(\Gamma, \mathbb{Z}) \arrow[d, "\cong"', "\phi"] & \Lambda^2 H^1(\Gamma, \mathbb{Z}) \arrow[l, "\cong"] \arrow[d, "\cong"', "\Lambda^2\phi"] \\
[\mathcal{L}] \in H^1(X, \mathcal{O}_X^*) \arrow[r, "c_1"] & H^2(X, \mathbb{Z}) & \Lambda^2 H^1(X, \mathbb{Z}) \arrow[l, "\cong"]
\end{tikzcd}$$

let's start from the image of c_1

It's easy to compute $\delta(\{e_\gamma\}_{\gamma\in\Gamma})$: take a lift $f_\gamma \in H$, ie $e^{2\pi i f_\gamma(z)} = e_\gamma(z)$. Then $\delta\{e_\gamma\} = F$, where

$$F(\gamma_1, \gamma_2) = f_{\gamma_2}(z+\gamma_1) - f_{\gamma_1+\gamma_2}(z) + f_{\gamma_1}(z) \in \mathbb{Z}$$

(✱)

The cup product induces an isomorphism

$$\begin{tikzcd}
\Lambda^2 H^1(\Gamma, \mathbb{Z}) \arrow[r, "\cong"'] & H^2(\Gamma, \mathbb{Z}) \\
E(\gamma_1, \gamma_2) = F(\gamma_1, \gamma_2) - F(\gamma_2, \gamma_1) & F \arrow[l, mapsto] \arrow[u, phantom, "\in"]
\end{tikzcd}$$

<u>Check</u> If we extend E $\mathbb{R}$-linearly to $V \times V \to \mathbb{R}$,

$$E(ix, iy) = E(x, y).$$

· kernel of G.

Def A Hermitian form is $H : V \times V \longrightarrow \mathbb{C}$ st
(i) H is $\mathbb{C}$-linear on the first factor
(ii) $H(z, w) = \overline{H(w, z)}$.

Def A Hermitian form is a Riemann form with respect to a lattice $\Gamma \subset V$ if
$$\operatorname{Im} H(\Gamma \times \Gamma) \subset \mathbb{Z}.$$

Eg $\Lambda = \mathbb{Z}\langle w_1 \rangle \oplus \mathbb{Z}\langle w_2 \rangle \subset \mathbb{C}$
$$H(z, w) = \frac{z\bar{w}}{\operatorname{Im}(w_1 \bar{w}_2)}.$$

Lemma There is a bijection
$$\left\{ \begin{array}{l} \text{Hermitian form} \\ H : V \times V \longrightarrow \mathbb{C} \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{l} \text{Real alternating form} \\ E : V \times V \to \mathbb{R},\ E(ix, iy) = E(x, y) \end{array} \right\}$$
$$H \longmapsto \operatorname{Im} H$$
$$E(ix, y) + iE(x, y) \longleftarrow\!\shortmid\ E$$